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A Classification of 4-Connected Graphs

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Having observed Tutte's classification of 3-connected graphs as those attainable from wheels by line addition and point splitting and Hedetniemi's classification of 2-connected graphs as those obtainable from K_3 by line addition, subdivision and point addition, one hopes to find operations which classify n-connected graphs as those obtainable from, for example, K_{n+1}. In this paper I give several generalizations of the above operations and use Halin's theorem to obtain two variations of Tutte's theorem as well as a classification of 4-connected graphs.

1. INTRODUCTION

All terms used are consistent with their definitions in [2]. In particular, "graph" shall mean a Michigan graph (i.e., undirected and without loops or multiple edges). Let $|G|$ denote the number of points in G; $u \in G$ means u is a point of G. Graph G is n-connected if the minimum number of points whose removal disconnects G or gives K_1 is $\geqslant n$; G is minimally-n-connected if G is n-connected but, for any edge e of G, G-e is not n-connected.

Menger's theorem states that the minimum number of points separating non-adjacent points u and v is the maximum number of disjoint u-v paths. Using this one proves Whitney's theorem, namely that G is n-connected if and only if for every pair of points u and v there are at least n point disjoint u-v paths. Let P_{uv} denote a path between u and v.

Two points u and v will be said to be n-connected if there are at least n point disjoint u-v paths. Thus for $|G| \geqslant n+2$ one shows that G is n-connected either by showing that the removal of any $n-1$ points of G leaves a connected graph or by showing that any two points are n-connected.

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2. Some Operations Preserving n-Connectivity

Let G be a graph, and let H be the graph obtained from G by adding a point u which we make adjacent to points $1, 2, \ldots, n-1$ and n of G. Then H will be said to be obtained from G by "point addition." It is clear that if G is n-connected and we add a point of degree at least n, then H is also n-connected. As corollaries of this one has the following two useful lemmas.

Lemma 1. *Let p and $u_1, u_2, \ldots, u_n$ be distinct points of an n-connected graph, then there are paths $P_{pu_1}, \ldots, P_{pu_n}$ where any two of these have only point p in common.*

Lemma 2. *Let $S_1 = \{u_1, \ldots, u_n\}$ and $S_2 = \{v_1, \ldots, v_n\}$ be disjoint sets of n points in an n-connected graph G. Then there are n paths, each from a u_i to a v_j, no two having a common point.*

I now give two generalizations of Tutte's concept of "(point) splitting." Let $u \in G$ with $\deg u \geqslant 2n-2$. Let H be the graph obtained by replacing u by two adjacent points, u_1 and u_2, and if x is adjacent to u in G, written x adj u, then make x adj u_1 or x adj u_2 (but not both), where we require that $\deg u_1 \geqslant n$ and $\deg u_2 \geqslant n$. (See Fig. 1a.) Let H be said to arise from

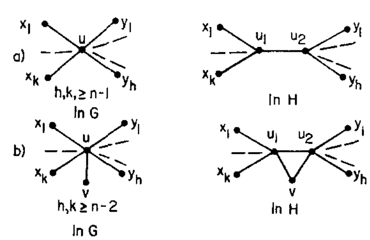

Fig. 1. "n-point-splitting" and "n-line-splitting."

G by "n-point-splitting." Let uv be an edge of G with $\deg u \geqslant 2n-3$ with u adjacent to $v, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_h$ where k and $h \geqslant n-2$. Let H be the graph obtained by replacing u by two adjacent points, u_1 and u_2, with v adj u_1, v adj u_2, u_1 adj x_i $(1 \leqslant i \leqslant k)$, and u_2 adj y_j $(1 \leqslant j \leqslant h)$. (See Fig. 1b.) Let H be said to arise from G by "n-line-splitting."

Theorem 0. *If G is n-connected and H arises from G by line addition, then H is n-connected.*

Theorem 1. *If G is n-connected and H arises from G by n-point-splitting, then H is n-connected.*

Proof. For $n = 1$ it is clear that 1-point-splitting leaves a connected graph. For $n \geqslant 2$, let $\bar{H}$ be a subgraph resulting from removing $n - 1$ points of H. If we remove u_1, u_2 and $n - 3$ others, then $\bar{H}$ is a subgraph of G resulting from the removal of $n - 2$ points, so it is connected. If we remove $S = \{u_1, w_2, ..., w_{n-1}\}$, then some y_i remains, say y_1. In $G - \{u, w_2, ..., w_{n-1}\}$ we can connect y_1 and any other point by some path. Since we can use the same paths in $H - \{u_1, w_2, ..., w_{n-1}\}$, and we have u_2 adj y_1, $\bar{H}$ is connected. Finally, if u_1 and u_2 are left in $\bar{H} = H - \{w_1, ..., w_{n-1}\}$, let $t \in \bar{H}(t \neq u_1$ or $u_2)$. Then $t \in G - \{w_1, ..., w_{n-1}\}$, so we have a path $P = t, ..., p, u$ in $G - \{w_1, ..., w_{n-1}\}$. If $p = x_i$ we have $t, ..., p, u_1$ in $\bar{H}$; if $p = y_j$ we have $t, ..., p, u_2, u_1$ in $\bar{H}$. Thus $\bar{H}$ is connected.

THEOREM 2. *If G is n-connected and H arises from G by n-line-splitting, then H is n-connected.*

Proof. For $n = 1$ it is clear that line splitting leaves a connected graph; for $n = 2$ it is clear that removing any one point from H leaves a connected graph. For $n \geqslant 3$ suppose deg $u \geqslant 2n - 2$, then k or h is at least $n - 1$, say $k \geqslant n - 1$. We can n-point-split u using $x_1, ..., x_k$ and $y_1, ..., y_h$, v. By Theorem 1 the resulting graph is n-connected; by Theorem 0 (adding line vu_1) H is n-connected. Now suppose deg $u = 2n - 3$, that is $k = n - 2 = h$. Remove any $n - 1$ points $\{w_1, ..., w_{n-1}\}$ of H and get graph $\bar{H}$. If we have not removed u_1 or u_2, then since there are $2n - 3$ points adjacent to u we have at least $2n - 3 - (n - 1) = n - 2 \geqslant 1$ points left adjacent to u_1 or u_2. Without loss of generality say y_1 is left. (For $n \geqslant 4$ some x_i or y_j is left; for $n = 3$ perhaps only v is left, but the proof remains the same.) Now y_1, u_2, u_1 connects y_1 to u_2 and u_1. Let t be any other point of $\bar{H}$. In $G - \{w_1, ..., w_{n-1}\}$ we have a path P_{tv_1}. Since we may replace any appearance of u by u_1u_2, u_2u_1, u_1 or u_2, as required, y_1 is connected to t in $\bar{H}$. Thus $\bar{H}$ is connected. If $\bar{H} = H - \{u_1, u_2, w_3, ..., w_{n-1})$ then $\bar{H} = G - \{u, w_3, ..., w_{n-1}\}$ is connected. If $\bar{H} = H - \{u_1, w_2, ..., w_{n-1}\}$, then u_2 had $n - 1$ points adjacent to it, so at least one remains, say y_1. (Again for $n = 3$, if we have to use v instead of y_1, the argument is the same.) Now y_1 adj u_2. Let $t \in \bar{H}$, then in $G - \{u, w_2, ..., w_{n-1}\}$ there is a path from t to y_1. The same path exists in $\bar{H}$. Thus $\bar{H}$ is connected.

THEOREM 3 (Halin [1]). *Every minimally-n-connected graph contains a point of degree n.*

THEOREM 4. *Suppose H is n-connected with $u \in H$ where* deg $u = n$ *and $|H| \geqslant n + 2$. If G is the graph obtained from H by deleting u and adding all edges between any two points adjacent to u, then G is also n-connected.*

Proof. Let $1, 2, \ldots, n$ denote the n points adjacent to u, then in G we have $\langle 1, 2, \ldots, n \rangle = K_n$ (where $\langle 1, 2, \ldots, n \rangle$ is the subgraph generated by the points $1, 2, \ldots, n$.) Let v and w be any two points of G, then in H we have n point disjoint paths from w to v. Suppose one of them contains u, say $P = w, \ldots, i, u, j, \ldots, v$ with $1 \leqslant i, j \leqslant n$. Since we have edge ij in G we can change P to $w, \ldots, i, j, \ldots, v$. Thus any two points are n-connected in G. So G is n-connected.

Now consider the classes of 1-connected, 2-connected, and 3-connected graphs.

THEOREM 5. *The class of minimally-*1*-connected graphs is the class of graphs obtained from K_2 by finite sequences of* 1*-point-splitting; the class of* 1*-connected graphs is the class of graphs obtained from K_2 by finite sequences of line addition and* 1*-point-splitting.*

Proof. Obvious—since the minimally-1-connected graphs are the trees.

THEOREM 6. *The class of* 2*-connected graphs is the class of graphs obtained from K_3 by finite sequences of line addition and* 2*-point-splitting.*

Proof. We induct on the number of points. Let H be a 2-connected graph with at least four points. Since we are allowing line addition, by Halin's theorem we may assume there is a point u of degree 2, say u adj 1 and u adj 2. Now by Theorem 4, $G = H - u + 12$ is 2-connected. By induction we can obtain G from K_3 with a finite sequence of line additions and 2-point-splittings. If H did not contain edge 12 (which would be true if we required H to be minimally-2-connected), then we can obtain H by 2-point-splitting point 1 in G; if H did contain edge 12, then we can 2-point-split and add a line to obtain H. Thus we can obtain all the 2-connected graphs. By Theorem 0 and Theorem 1, we only obtain 2-connected graphs.

THEOREM 7. *The class of* 3*-connected graphs is the class of graphs obtained from K_4 by finite sequences of line addition,* 3*-point-splitting and* 3*-line-splitting.*

Proof. By Theorems 0, 1, and 2 all one obtains is 3-connected graphs. To show that one can obtain all of the 3-connected graphs it suffices to observe that all of the wheels can be obtained from K_4 by finite sequences of 3-line-splittings. Tutte's theorem tells us we can then obtain all 3-connected graphs from the wheels by line addition and 3-point-splitting. An alternate proof ([4]) is outlined here. Induct on the number of points in the graph. Let H be a 3-connected graph with at least five points. By

Halin's theorem, since we allow line addition we may assume that H has a point of degree three, say deg $u = 3$ with u adj 1, u adj 2 and u adj 3. We then consider the four cases for $\langle 1, 2, 3\rangle$ (see Fig. 2). For Cases 1,

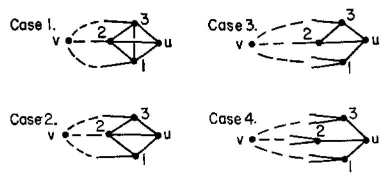

FIG. 2. Structure at a point of degree three in a 3-connected graph.

2, and 3 we can obtain G as in Theorem 4 and use 3-line splitting and line addition, 3-line-splitting, and 3-point-splitting, respectively. For Case 4, we let H_{1u} be the graph obtained from H by identifying points 1 and u and similarly for H_{2u} and H_{3u}. It can be shown that at least one of H_{1u}, H_{2u} and H_{3u} will be 3-connected whenever H is. By 3-point-splitting the appropriate H_{iu} we obtain H.

It should be noted that if H is minimally-3-connected then Case 1 cannot occur.

It is interesting to consider what lines can be removed from $\langle 1, 2,\ldots, n\rangle$ when one adds a point of degree n to these n points if n-connectivity is to be preserved. In view of the next theorem it is especially interesting when $n = 3$.

Let 1, 2 and 3 be arbitrary points in a graph G, and let H be the graph that comes from G by deleting all the lines in $\langle 1, 2, 3\rangle$, adding a point u adjacent to 1, 2 and 3, and then adding any set L of lines which keeps $\deg i \geqslant 3$, where $1 \leqslant i \leqslant 3$, and where L is a subset of $\{12, 13, 23\}$. Graph H will be said to be obtained from G by "3-soldering." (Equivalently, 3-soldering consists of adding a point u adjacent to points 1, 2 and 3, and adding all lines between any two points adjacent to u, and then possibly removing a line from $\langle 1, 2, 3\rangle$ which connects two points of degree at least 4, possibly removing a second line under the same restriction, and finally possibly removing the third line if the end points still have degree 4.)

THEOREM 8. *The class of* 3-*connected graphs is the class of graphs obtained from* K_4 *by finite sequences of line addition and* 3-*soldering.*

Proof. Let H be any 3-connected graph where, as before, we may assume H has a point of degree 3. Again, induct on the number of points in H. If H has more than four points, let G be as in Theorem 4. Clearly we can 3-solder G to obtain H. Conversely it must be shown that 3-soldering preserves 3-connectivity. Let G be 3-connected and suppose that H comes from G by 3-soldering where we have added point u adjacent to points 1, 2 and 3. Note that H has at least five points u, 1, 2, 3 and v, and we have paths P_{v1}, P_{v2} and P_{v3} as in Lemma 1. Let S be any two point set in H. If $u \notin S$ then 1 adj u, 2 adj u, 3 adj u, and if w is any other point in $H - S$, then at least one of the paths P_{w1}, P_{w2} and P_{w3} (from Lemma 1) remains, and thus we have a path from w to u. Consequently $H - S$ would be connected. Suppose $u \in S$. Consider four cases for H, as in Figure 2.

Case 1. H contains the three edges 12, 23 and 13. Since G is 3-connected, H-u is three connected. Hence H-S is connected.

Case 2. H contains two of the three lines, say 12 and 23. If $S = \{u, 1\}$ then 3 adj 2 and for any w in H-S, P_{w2} remains. So H-S is connected, and similarly for $S = \{u, 3\}$. If $S = \{u, 2\}$, then P_{v1} and P_{v3} connect 1 and 3, and, for any w in H-S, P_{w3} remains. So H-S is connected. If $S = \{u, w\}$ with $w \notin \{1, 2, 3\}$ then $P = 1, 2, 3$ connects points 1, 2 and 3. If t is any point in H-S then at least one of P_{t1}, P_{t2} and P_{t3} remains. So H-S is connected.

Case 3. H contains one of the three lines, say 23. If $S = \{u, 1\}$ then for every $t \in H$-S, P_{t2} remains. So H-S is connected. If $S = \{u, 2\}$, then P_{v1} and P_{v3} connect 1 and 3, and, for any other w in H-S, P_{w3} remains. So H-S is connected, and similarly for $S = \{u, 3\}$. If $S = \{u, w\}$ with $w \neq 1$, 2 or 3, we let s_1 and s_2 be the points adjacent to 1 in H-u. At least one of these is in H-S, say s_1. Then either $P_{s_1 2}$ or $P_{s_1 3}$ remains, so 1 is connected to 2 and 3. Now for any other t in H-S either P_{t2} or P_{t3} remains, so t is connected to 2. Thus H-S is connected.

Case 4. H contains none of the three lines. If $S = \{u, 1\}$, then P_{v2} and P_{v3} connects 2 and 3 and for any other t in H-S, P_{t2} remains. Thus H-S is connected, and similarly for $S = \{u, 2\}$ or $\{u, 3\}$. Suppose $S = \{u, w\}$ with $w \notin \{1, 2, 3\}$. Let s_1 and s_2 be adjacent to 1 in H-u, then at least one is in H-S, say s_1. Now either $P_{s_1 2}$ or $P_{s_1 3}$ remains, say $P_{s_1 2}$. Then 1 and 2 are connected. Now similarly 3 is connected to at least one of the points 1 and 2. Thus 1, 2, and 3 are connected in H-S. Now for any other t in H-S, at least one of P_{t1}, P_{t2} and P_{t3} remains. Thus H-S is connected.

Consequently, if H comes from G by 3-soldering then G is 3-connected implies H is 3-connected.

3. 4-Connected Graphs

With a point u of degree four there are 11 possible graphs on the four points adjacent to it (see Fig. 4), as opposed to the four graphs adjacent to a point of degree three (see Fig. 2). Theorem 8 essentially says that with a 3-connected graph H that contains a point of degree three one can use Theorem 4 to obtain a 3-connected graph G with one fewer points, and then one can use 3-soldering (which always preserves 3-connectivity) to obtain H from G—thus setting the basis for an induction argument. Figure 3 shows that Theorem 8 can not have a counterpart for 4-connected

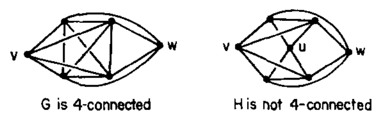

Fig. 3. An attempted "4-soldering."

graphs. However, if a further restriction is put on the set of lines that must be added among the points adjacent to u, namely that it must contain two lines incident with the same point, then eight of the 11 cases can be handled.

Let 1, 2, 3 and 4 be arbitrary points in a graph G, and let H be the graph that comes from G by deleting all the lines in $\langle 1, 2, 3, 4\rangle$, adding a point u adjacent to 1, 2, 3 and 4, and then adding any set L of lines from $\{12, 13, 14, 23, 24, 34\}$ which contains two lines incident with the same point and which makes $\deg i \geq 4$, where $1 \leq i \leq 4$. Graph H will be said to be obtained from G by "4-soldering." (Equivalently, 4-soldering consists of adding a point u adjacent to points 1, 2, 3, and 4 and adding all lines between any two points adjacent to u, and then possibly removing a line from $\langle 1, 2, 3, 4\rangle$ which connects two points of degree at least five and whose removal will leave a $K_{1,2}$ in $\langle 1, 2, 3, 4\rangle$, etc.)

Theorem 9. *4-soldering preserves 4-connectivity.*

Proof. Let K be a 4-connected graph, and let H be obtained from K by 4-soldering where we have added point u adjacent to 1, 2, 3, and 4. Examine the eight possibilities for H (namely, Case 1–8 in Fig. 4), and let G be the graph obtained from H as in Theorem 4 (see Fig. 4). Since G is obtained from K by line addition, G will be 4-connected. Hence it suffices to show that G is 4-connected implies that H is 4-connected. Let v denote a fifth point of G, then from Lemma 1 one obtains P_{v1}, P_{v2}, P_{v3}, and P_{v4}. Let S be a three point set in H. Suppose $u \notin S$. If $1 \leq i \leq 4$ and

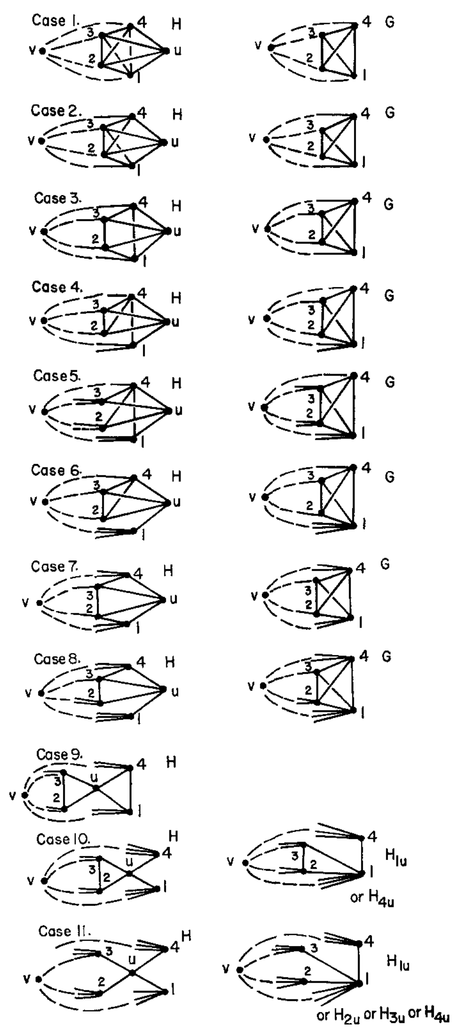

FIG. 4. Structure at a point of degree four in 4-connected graph H.

$i \in H\text{-}S$ then i adj u, and for any other point w in $H\text{-}S$ at least one of P_{w1}, P_{w2}, P_{w3}, and P_{w4} remains, which implies that w is connected to u in $H\text{-}S$. Thus $H\text{-}S$ is connected. Suppose that $u \in S$. If $w \in H\text{-}S$ and $w \neq i$, where $1 \leqslant i \leqslant 4$, then at least two of P_{w1}, P_{w2}, P_{w3}, and P_{w4} remain. Hence to show that $H\text{-}S$ is connected it will suffice to show that the points from $\{1, 2, 3, 4\}$ in $H\text{-}S$ are connected. If S is u and two of the points adjacent to it, then the other two are connected by a path through v. Now w_1 and w_2 will represent points which are not 1, 2, 3 or 4.

Case 1. G is 4-connected and H is obtained from G by "point adding of degree 4." Thus H is 4-connected.

Case 2. If $S = \{u, w_1, w_2\}$ then 1, 2, 3, and 4 are clearly connected. If $S = \{u, i, w_1\}$ then the other three points adjacent to u are clearly connected. Hence H-S is always connected.

Case 3. If $S = \{u, w_1, w_2\}$ or $\{u, i, w_1\}$ then it is clear that H-S is connected. Hence H-S is always connected.

Case 4. Since point 1 must have degree at least five in G, one can 4-line-split edge 14 to obtain H. Thus H is 4-connected by Theorem 2.

Case 5. If $4 \notin S$ and $i \in H$-S, where $1 \leqslant i \leqslant 3$, then i adj 4. If $S = \{u, 4, w_1\}$, then since degree $i \geqslant 5$ in G, where $1 \leqslant i \leqslant 3$, then there are points s, t and r (not necessarily distinct) adjacent to 1, 2 and 3, respectively, in H-S. Now P_{s2} or P_{s3} remains in H-S, say P_{s2}. Hence 1 and 2 are connected. Now either P_{r2} or P_{r1} remains. Hence 1, 2, and 3 are connected. Hence H-S is always connected.

Case 6. Since deg $1 \geqslant 6$ in G one can 4-point-split point 1 to get H. Hence H is 4-connected by Theorem 1.

Case 7. If $S = \{u, w_1, w_2\}$ then clearly 1, 2, 3, and 4 are connected. If $S = \{u, 1, w_1\}$ then 2, 3, and 4 are connected, and similarly for $\{u, 4, w_1\}$. If $S = \{u, 2, w_1\}$ then since deg $1 \geqslant 5$ in G there is a point s adj 1 in H-S, and now either P_{s3} or P_{s4} remains, so 1 is connected to 3 and 4. A similar argument holds for $\{u, 3, w_1\}$. Hence H-S is always connected.

Case 8. If $S = \{u, w_1, w_2\}$, then since deg $1 \geqslant 6$ in G there remains a point s in H-S with s adj 1, and at least one of P_{s2}, P_{s3}, and P_{s4} remains. If $S = \{u, 1, w_1\}$ then 2, 3, and 4 are connected. If $S = \{u, 2, w_1\}$ then one still has s in H-S with s adj 1, and either P_{s3} or P_{s4} remains. Thus 1, 3, and 4 are connected, and similarly for $\{u, 4, w_1\}$. Finally for $S = \{u, 3, w_1\}$, since deg $1 \geqslant 6$ and deg $2 \geqslant 5$ and deg $4 \geqslant 5$ in G there are points s, t and r in H-S adjacent to 1, 2, and 4, respectively. As in Case 5, 1, 2, and 4 are connected. Hence H-S is always connected.

Consequently, H is 4-connected whenever G is 4-connected, and the theorem is proved.

I now show that if H is 4-connected with a point u of degree 4 from Case 10 or 11 then there is a 4-connected graph G from which we can obtain H by 4-point-splitting. Recall that H_{uv} denotes the graph obtained from H by identifying points u and v. Now the three points "$4u$" and x and y separate H_{4u} if and only if the four points 4, u, x, and y separate H.

Suppose a graph H is n-connected and $H - \{u_1, u_2, \ldots, u_n\}$ is disconnected where A is one of the components. If $a \in A$ the point disjoint paths $P_{au_1}, P_{au_2}, \ldots, P_{au_n}$ can not contain a point from another component. Even though each $u_i \notin A$, the paths will be referred to as paths "in A" to $u_1, u_2, \ldots, u_n$.

THEOREM 10. *If H is 4-connected with a point u of degree four that is adjacent to* 1, 2, 3 *and* 4, *as in Case* 10, *then either H_{4u} or H_{1u} is 4-connected.*

Proof. Suppose $S = \{w_1, w_2, w_3\}$ separates H_{4u}, then it is clear that $4 \in S$ (let "4" be the point $4u$), $S \neq \{4, 1, 2\}, \{4, 1, 3\}, \{4, 2, 3\}$, or $\{4, 1, w_1\}$ where $w_1 \neq 2$ or 3. It remains to be shown that when H_{4u} (respectively H_{1u}) can be separated by removing point 4 (respectively 1) and two others, then H_{1u} (respectively H_{4u}) will be 4-connected. As noted, the other two points can not be 2 and 3. The first three claims consider the case where one point from $\{2, 3\}$ is selected. In what follows, w_1, w_2, v_1, and v_2 shall represent points other than 1, 2, 3 and 4.

Claim 1. If $\{4, 2, w_1\}$ separates H_{4u}, then H_{1u} is 4-connected.

Proof. Suppose H is disconnected by $S = \{4, u, 2, w_1\}$. For any point v either P_{v3} or P_{v1} remains, and so we have exactly two components, say A_{4u} and B_{4u} with $1 \in A_{4u}$ and $3 \in B_{4u}$. Since $\{4, u, 2, w_1\}$ separates H, Lemma 1 implies any $a \in A_{4u}$ has point disjoint paths P_{a4}, P_{a1}, P_{a2}, and P_{aw_1} in A_{4u}, and any $b \in B_{4u}$ has point disjoint paths P_{b4}, P_{b3}, P_{b2}, and P_{bw_1} in B_{4u}.

Let S_1 be a four point set of H containing 1 and u. It will be shown that $H\text{-}S_1$ is connected. If $4 \in S_1$ then clearly $H\text{-}S_1$ is connected. If $S_1 = \{1, u, 2, w_1\}$ then 4 is connected to any other point in $H\text{-}S_1$. If $S_1 = \{1, u, 2, w\}$ with $w \in B_{4u}$ then any point in A_{4u} is connected to 4 and w_1 (there are other points in A_{4u} besides point 1 because deg $1 \geqslant 4$ and 1 is not adjacent to 4), and any other point t left in B_{4u} has either P_{t4} or P_{tw_1} in $H\text{-}S_1$. (A similar case is $S_1 = \{1, u, 2, w\}$ with $w \in A_{4u}$.) If $S_1 = \{1, u, w_1, w\}$ with $w \in B_{4u}$ then $a \in A_{4u}$ is connected to 2 and 4, and any other point t in B_{4u} has P_{t2} or P_{t4} in $H\text{-}S_1$. (A similar case is $S_1 = \{1, u, w_1, w\}$ with $w \in A_{4u}$.) If $S_1 = \{1, u, p_1, p_2\}$ with p_1 and p_2 in B_{4u} then any a in A_{4u} is connected to 4, 2, and w_1, and any other t in B_{4u} has at least one of P_{tw_1}, P_{t2}, and P_{t4} left in $H\text{-}S_1$. (A similar argument holds for p_1 and p_2 in A_{4u}.) Suppose $S_1 = \{1, u, 3, a\}$ with $a \in A_{4u}$ then for point b in $B_{4u} - \{3\}$ we have P_{bw_1}, P_{b2}, and P_{b4}, and for any other point t in A_{4u} at least two of P_{tw_1}, P_{t2}, and P_{t4} are in $H\text{-}S_1$. Finally if $S_1 = \{1, u, a, b\}$ with a in A_{4u} and $3 \neq b$ with b in B_{4u} then 3 adj 2 and 3 is connected in $H\text{-}S$ to w_1 or 4, say w_1. Since deg $4 \geqslant 4$ and 4 is not adjacent to 1 there is a point q in $H\text{-}S_1$ with q adj 4. Whether $q \in A_{4u}$ or $q \in B_{4u}$

at least one of P_{qw_1} and P_{q2} is in H-S_1. Thus 4, 2 and w_1 are connected, and for any point t of H-S_1 at least one of P_{t2}, P_{t4} and P_{tw_1} is in H-S_1.

Similarly one can show the following:

Claim 2. If $\{4, 3, w_1\}$ separates H_{4u}, then H_{1u} is 4-connected.

Claim 3. If $\{1, 2, w_1\}$ or $\{1, 3, w_1\}$ separates H_{1u}, then H_{4u} is 4-connected.

Now if H_{4u} and H_{1u} are both not 4-connected then $\{4, u, w_1, w_2\}$ separates H and $\{1, u, v_1, v_2\}$ separates H.

Claim 4. $\{w_1, w_2\} \neq \{v_1, v_2\}$.

Proof. There is a path P from 1 to 3 in $H - \{u, w_1, w_2\}$. If $4 \notin P$ then $\{4, u, w_1, w_2\}$ does not separate H; if $4 \in P$ then $\{1, u, w_1, w_2\}$ does not separate H.

Claim 5. $w_1 \neq v_1$.

Proof. Suppose $w_1 = v_1$. Let A_{4u} and B_{4u} be the components of $H - \{4, u, w_1, w_2\}$ with 1 in A_{4u}. We have paths P_{34}, P_{3u}, P_{3w_1}, P_{3w_2} in B_{4u}. Thus v_2 must be on P_{34}, which implies that v_2 is in B_{4u}. Now deg $1 \geqslant 4$ and 1 not adjacent to 4 implies there is a point p adjacent to 1 with $p \in A_{4u}$. Then we have P_{p4}, P_{p1}, P_{pw_1}, P_{pw_2} in A_{4u}. Now in $H - \{1, u, v_1, v_2\}$ we have P_{p4}, P_{pw_2}, and P_{3w_2}. But this means 4 and 3 and 2 are connected, and so $\{1, u, v_1, v_2\}$ does not separate H, contrary to its definition.

Claim 6. $\{w_1, w_2\}$ and $\{v_1, v_2\}$ cannot be disjoint.

Proof. As in Claim 5, one of v_1 and v_2 is on P_{34}, say v_2. Let p adj 1 with $p \in A_{4u}$. Let A_{1u} and B_{1u} be the components of $H - \{1, u, v_1, v_2\}$ with $3 \in B_{1u}$, $4 \in A_{1u}$. Since we have P_{34}, p_{3u}, P_{3w_1}, P_{3w_2} in B_{4u} and P_{p4}, P_{p1}, P_{pw_1}, P_{pw_2} in A_{4u}, 3 and p cannot be separated by $\{1, u, v_1, v_2\}$. So $p = v_1$ or p is in B_{1u}. Since deg $4 \geqslant 4$ there is a point q adj 4 where $q \neq u$ and q is not on P_{p4} or P_{34}. If $q \in A_{4u}$ then one of P_{qw_1} and P_{qw_2} does not contain p. Thus we have a walk 4, q,..., w_1,..., 3 or 4, q,..., w_2,..., 3 in $H - \{1, u, p, v_2\}$. If $q \in B_{4u}$ then one of P_{qw_1} and P_{qw_2} does not contain v_2. Thus we have a walk 4, q,..., w_1,..., 3 or 4, q,..., w_2,..., 3 in $H - \{1, u, p, v_2\}$. Therefore $p \neq v_1$. So suppose p is in B_{1u}, and then v_1 must be on P_{p4}. Again we consider point q. If q is in A_{4u} then one of P_{qw_1} and P_{qw_2} does not contain v_1; if q is in B_{4u} then one of P_{qw_1} and P_{qw_2} does not contain v_2. But then in either case we have a walk in $H - \{1, u, v_1, v_2\}$ from 4 to 3. But this means 4 and 3 and 2 are connected, and so $\{1, u, v_1, v_2\}$ does not separate H.

Consequently, the theorem is proved.

THEOREM 11. *If H is 4-connected with a point u of degree four that is adjacent to* 1, 2, 3 *and* 4, *as in Case* 11, *then at least one of* H_{1u}, H_{2u}, H_{3u}, *and* H_{4u} *is 4-connected.*

Proof. It will be shown that if some H_{iu} is not 4-connected ($1 \leqslant i \leqslant 4$) then at least one H_{ju} ($1 \leqslant j \leqslant 4, j \neq i$) is 4-connected.

Any three point set which separates H_{4u} must contain point 4. Consider a four point set S_4 that contains 4 and u and separates H. Then $S_4 \neq \{4, u, 1, 2\}$ or $\{4, u, 1, 3\}$ or $\{4, u, 2, 3\}$. First it is assumed that the only point of S_4 that is adjacent to u is point 4. If $S_4 = \{4, u, w_1, w_2\}$ with $w_i \neq 1, 2$ or 3 then for each point t in H-S_4 at least one of P_{t1}, P_{t2} and P_{t3} (using Lemma 1 with the four points 1, 2, 3, and 4) is in H-S_4. Thus there are at most three components.

Claim. If H-S_4 has three components A, B and C with $1 \in A$, $2 \in B$, and $3 \in C$, then H_{1u} is 4-connected.

Proof. Since 1, 2 and 3 are not adjacent to 4, there are points p_1 adj 1, p_2 adj 2, and p_3 adj 3 with $p_1 \in A$, $p_2 \in B$ and $p_3 \in C$. Let v_1 and v_2 be points other than w_1, w_2 and 4. If $S_1 = \{1, u, v_1, v_2\}$ then at least one component does not contain v_1 or v_2, say p_i is in that component. Then p_i has paths to w_1, w_2, and 4 in H-S_1, as does any point in that component. If $t \in H$-S_1 is in one of the other components then at least one of P_{t4}, P_{tw_1}, and P_{tw_2} is in H-S_1. Thus H-S_1 is connected in this case. If S_1 contains 1, u and two other points in S_4 then the remaining point in S_4 is clearly connected to every point in H-S_1. If $S_1 = \{1, u, 4, v_1\}$ then w_1 and w_2 are both connected to every point in the two components not containing v_1 (hence w_1 and w_2 are connected), and every point in the component containing v_1 is connected to at least one of w_1 and w_2. Cases $S_1 = \{1, u, w_1, v_1\}$ or $\{1, u, w_2, v_1\}$ are handled similarly.

Claim. If H-S_4 has only two components A and B with $1 \in A$ and $2 \in B$ and $3 \in B$, then H_{1u} is 4-connected.

Proof. For $t \in A$ ($t \in B$) let P_{t4}, P_{tu}, P_{tw_1}, and P_{tw_2} denote the paths in A (in B) from Lemma 1. If S_1 contains 1, u and two other points of S_4 then the remaining point in S_4 is connected to every point in H-S_1. If $S_1 = \{1, u, w_1, b\}$ with $b \in B$ then for each point t in A we have P_{t4} and P_{tw_2} in H-S_1, and if $t \in H$-S_1 is in B at least one of P_{t4} and P_{tw_2} is in H-S_1. A similar case is $S_1 = \{1, u, w_1, a\}$ with $a \in A$. Similar cases are $S_1 = \{1, u, w_2, t\}$ and $\{1, u, 4, t\}$ where t may be in A or B. If $S_1 = \{1, u, b_1, b_2\}$ with $b_1, b_2 \in B$ then, for each t in A, P_{t4}, P_{tw_1}, and P_{tw_2} are in H-S_1, and if $t \in H$-S_1 is in B at least one of P_{t4}, P_{tw_1}, and P_{tw_2} is in H-S_1, and similarly for $S_1 = \{1, u, a_1, a_2\}$ with $a_1, a_2 \in A$. Now suppose

$S_1 = \{1, u, a, b\}$ with $a \in A$ and $b \in B$, then $b \neq 2$ or $b \neq 3$, say $b \neq 2$. Suppose $b \in P_{24}$. Note that 2 and 4 are connected in $H\text{-}S_1$ and that 2 is connected to at least one w_i, say w_1. Now if $b \neq 3$ then 3 is connected to at least one of the points 4 and w_1. Consequently, 2 and 4 and 3 (if $b \neq 3$) are connected in $H\text{-}S_1$. Thus $H\text{-}S_1$ would be connected if $b \notin P_{24}$. Suppose $b \in P_{24}$, then P_{2w_1} and P_{2w_2} are in $H\text{-}S_1$. Now 4 adj q with $q \neq u$ and q not on P_{24} or P_{a4}. If $q \in B$ then one of P_{qw_1} and P_{qw_2} does not contain b, and so we have a walk 4, $q,\ldots, w_1$ (or w_2),..., 2 in $H\text{-}S_1$; if $q \in A$ then one of P_{qw_1} and P_{qw_2} does not contain a, and so we have a walk 4, $q,\ldots, w_1$ (or w_2),..., 2 in $H\text{-}S_1$. As above, this implies that $H\text{-}S_1$ is connected.

Consequently the claim is true and we now have that if any $S = \{i, u, w_1, w_2\}$ separates H where $1 \leqslant i \leqslant 4$ and w_1, w_2 are distinct from 1, 2, 3 and 4 then at least one of H_{1u}, H_{2u}, H_{3u}, and H_{4u} is 4-connected. For the second part it is assumed that H cannot be separated by a four point set containing u and exactly one point adjacent to it. It is assumed that there is a four point separating set of H containing u and two points adjacent to it.

Claim. If no such $S = \{i, u, w_1, w_2\}$ separates H, and $S_4 = \{4, u, 2, w_1\}$ separates H with $1 \in A$ and $3 \in B$, then H_{1u} is 4-connected.

Proof. For $t \in A$ ($t \in B$) let P_{t4}, P_{tu}, P_{t2} and P_{tw_1} denote the paths in A (in B) from Lemma 1. By assumption $S_1 = \{1, u, v_1, v_2\}$ does not disconnect. If $S_1 = \{1, u, 2, w_1\}$ then 4 is clearly connected to every point of $H\text{-}S_1$. If $S_1 = \{1, u, 2, v\}$ and $v \in B$ we have P_{p_14} and $P_{p_1w_1}$ in $H\text{-}S_1$ where p_1 adj 1 is in A, and if $t \in H\text{-}S_1$ is in B then at least one of P_{tw_1} and P_{t4} is in $H\text{-}S_1$. Similarly for $v \in A$ we get $H\text{-}S_1$ is connected. Similar cases are $S_1 = \{1, u, 4, w_1\}$ or $S_1 = \{1, u, 4, v\}$. Finally if $S_1 = \{1, u, 3, v\}$ where $v = w_1$ or $v \in B$ then P_{p_14} and P_{p_12} connect 2 and 4, and so $H\text{-}S_1$ is connected; and if $S_1 = \{1, u, 3, a\}$ with $a \in A$ then P_{p_34} and P_{p_32} connect 2 and 4 where p_3 adj 3 is in B, and so $H\text{-}S_1$ is connected.

Since similar claims hold for $S_4 = \{4, u, 1, w_1\}$ and $S_4 = \{4, u, 3, w_1\}$, the theorem is proved.

In Fig. 5 we have an example of a 4-connected graph H which can not be obtained from any other 4-connected graph G by line addition, 4-

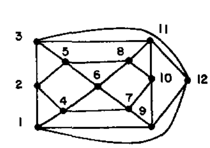

FIG. 5. An "unreachable" graph.

soldering, 4-point-splitting, or 4-line-splittting. The author has been referred to [6] in which D. Barnette generates the duals of the planar 4-connected graphs by "face splittings." Surprisingly the dual of the graph in Fig. 5 is used there to show the necessity of using "subdivisions of hexagons." Barnette's approach is particularly pleasing in that the "vertex splitting" duals of his operations will generate the planar 4-connected graphs from $\bar{K}_2 + C_4$, and one does not need solderings or line additions.

The following definition can be given in greater generality [4], but this will suffice to characterize the 4-connected graphs. Suppose $p \in G$ with $\deg p \geqslant 6$. Let H be the graph obtained from G by replacing p by a K_3, say on points p_1, p_2 and p_3, and for each point v of G with v adj p we make v adjacent to exactly one p_i, where we insure that $\deg p_i \geqslant 4$. Let H be said to be obtained from G by "3-fold–4-point-splitting." (See Fig. 6.)

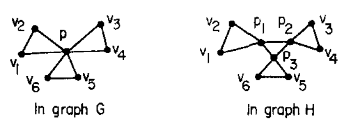

FIG. 6. An example of 3-fold–4-point splitting.

THEOREM 12. *If G is 4-connected and H is obtained from G by 3-fold–4-point-splitting, then H is 4-connected.*

Proof. Suppose p_1 adj v_1, p_1 adj v_2, p_2 adj v_3, p_2 adj v_4, p_3 adj v_5, and p_3 adj v_6 in H. If $S = \{p_1, p_2, p_3\}$ then H-S is G-p, so H-S is connected. If S has exactly two of p_1, p_2 and p_3, say $S = \{p_2, p_3, w\}$, then either v_1 or v_2 is in H-S, say v_1, Now p_1 adj v_1 and if t is any other point in H-S then the path from t to v_1 in $G - \{p, w\}$ is also in H-S. Thus H-S is connected. If S has exactly one of p_1, p_2, and p_3, say $S = \{p_3, w_1, w_2\}$, then one of v_1, v_2, v_3 and v_4 remains, say v_1. Now p_2, p_1, v_1 connects p_1 and p_2 to v_1 in H-S, and if t is any other point in H-S then the path from t to v_1 in $G - \{p, w_1, w_2\}$ is also in H-S. Thus H-S is connected. Finally, if $S = \{w_1, w_2, w_3\}$ does not contain a p_i then p_1, p_2, and p_3 are connected and if $t \in H$-S, then $t \in G$-S. So there is a path in G-S from t to p, and this implies there is a path in H-S to a p_i, $1 \leqslant i \leqslant 3$. Thus H-S is connected.

LEMMA 3. *If* 4-*connected graph H is as in Fig.* 6 *with* $\deg p_1 = \deg p_2 = \deg p_3 = 4$ *and with* v_1 adj v_2, v_3 adj v_4 *and* v_5 adj v_6, *then G is* 4-*connected, where G is obtained from H by identifying* p_1, p_2, *and* p_3 *as a single point p.*

Proof. If $S = \{w_1, w_2, w_3\}$ does not contain p then G-S is clearly connected. Suppose $S = \{p, w_1, w_2\}$. At least one of the three adjacent pairs remains, say v_1 and v_2. Suppose $v_3 \in G$-S. In H there are four v_1-v_3 paths P_1, P_2, P_3, and P_4. If any P_i contains only one of p_1, p_2 and p_3 then, since we have edges v_1v_2, v_3v_4, and v_5v_6, we can use these edges to replace the respective p_i in $v_1p_1v_2$, $v_4p_2v_3$, and $v_5p_3v_6$ (or $v_6p_3v_5$). Now at most one P_i can contain two of p_1, p_2 and p_3. Consequently we have three point disjoint v_1-v_3 paths in H which do not contain a p_i, $1 \leqslant i \leqslant 3$. Thus at least one of the paths in H will give us a v_1v_3 path in G-S. Similarly if $v_i \in G$-S, $4 \leqslant i \leqslant 6$, then v_1 and v_i are connected in G-S. Now if $w \in G$-S and $w \neq v_i$ then w has point disjoint paths in H to v_1, v_2, p_2 and p_3. Consequently in G-S there is a path from w to one of the points v_i, $1 \leqslant i \leqslant 6$. Thus G-S is connected.

LEMMA 4. *Suppose a* 4-*connected graph H is as in Fig.* 7 *with* deg deg u = deg 4 = 4 *where u and* 4 *are "Case* 9 *points" (Fig. 4) and* deg $1 \geqslant 5$, *then the graph G obtained from H by identifying points* 4 *and u is* 4-*connected.*

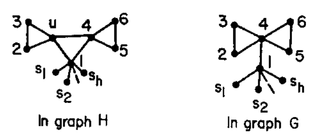

FIG. 7. Identifying adjacent Case 9 points.

Proof. Let s_1, s_2, s_3 be three points adjacent to 1 other than u and 4. Since u and 4 are Case 9 points, s_i is not 2, 3, 5 or 6. I will show that any two points of G are 4-connected.

4 *and* 1: Apply Lemma 2 to $\{1, s_1, s_2, s_3\}$ and $\{u, 2, 3, 4\}$ in H to get paths P_1, P_2, P_3 and P_4. One path, say P_1, must be 1, u; say P_2 takes s_1 to 2, P_3 takes s_2 to 3, and P_4 takes s_3 to 4 (we may assume P_4 has only one of points 5 and 6, say 5). Now in G we have 1, 4 and 1, s_1,..., 2, 4, and 1, s_2,..., 3, 4 and 1, s_3,..., 5, 4.

4 *and w where $w \neq$* 1: In H, 4 and w are 4-connected, say by P_1 = 4, 1,..., w and P_2 = 4, 5,..., w and P_3 = 4, 6,..., w and P_4 = 4, u, 2 (or 3),..., w. One need only change P_4 to 4, 2 (or 3),..., w in G.

1 *and w where $w \neq$* 4: Using Lemma 2 we can get point disjoint paths P_1, P_2, P_3 and P_4 from 1, s_1, s_2 and s_3 to p_1, p_2, p_3 and p_4 respectively where p_i adj w, $1 \leqslant i \leqslant 4$. (If, for example, $s_1 = p_2$ then we let P_2 be a path with only one point and we could apply Lemma 2 to $\{1, s_2, s_3\}$ and $\{p_1, p_3, p_4\}$ in the 3-connected graph H-p_2.) Suppose P_1 contains

u or 4, say u (we can assume it does not contain both). If no P_i contains point 4, one simply substitutes 4 for u in G, but if one does, say $P_2 = s_1, ..., 5, 4, 6, ..., p_2$ then we can use edge 56 and obtain $\bar{P}_2 = s_1, ..., 5, 6, ..., p_2$ and proceed as above. Now suppose P_1 does not contain 4 or u. If any P_i contains 4 and u we can assume they are adjacent and use point 4 in G in place of points u and 4 in H. If any P_i contains just 4 or just u not as an end point then we can use edge 56 or 23, respectively. (Note that at most one of u and 4 can be a p_i.) Thus in G we can find $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$, $\bar{P}_4$ for $\{1, s_1, s_2, s_3\}$ and $\{p_1, p_2, p_3, p_4\}$, and so 1 and w are 4-connected in G.

w_1 and w_2 with $w_i \neq 4$ or 1: We have P_1, P_2, P_3, and P_4 which 4-connect w_1 and w_2 in H. Suppose only one contains 4 or u. If it contains both we may assume they are adjacent and use point 4 in G; if it contains only one then in G we can use 23 for 2, u, 3 or 3, u, 2 and edge 56 for 5, 4, 6 or 6, 4, 5, and 1, 4, 2 for 1, u, 2, etc. Suppose $u \in P_1$ and $4 \in P_2$. Then one of them does not contain point 1, say P_2. We can use edge 56 in P_2 and proceed as above.

LEMMA 5. *If u is a Case* 9 *point of degree* 4 *in a minimally*-4-*connected graph H* (*see Fig. 8*), *then* deg 4 = 4 *or* deg 1 = 4.

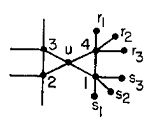

FIG. 8. Structure not possible at a point u of degree four in a minimally-4-connected graph.

Proof. Suppose deg $1 \geqslant 5$ and deg $4 \geqslant 5$. Label the points adjacent to points 1 and 4 as $s_1, s_2, s_3, ...$ and $r_1, r_2, r_3, ...$, respectively, such that if 1 and 4 are both adjacent to $p_1, p_2, ..., p_k$ (points other than u) then $s_1 = r_1 = p_1, ..., s_k = r_k = p_k$. Now H-p_1 is 3-connected, $H - \{p_1, p_2\}$ is 2-connected and $H - \{p_1, p_2, p_3\}$ is 1-connected, and so by the appropriate application of Lemma 2 we have point disjoint paths P_1, P_2, P_3 and P_4 taking s_1, s_2, s_3 and 1, respectively, to $\{r_1, r_2, r_3, 4\}$. If $P_4 \neq 1, 4$, then we have four point disjoint 1-4 paths not using edge 14. But then H-14 is 4-connected, and this would be a contradiction. So we can assume $P_4 = 1, 4$. Suppose u is on one of the paths, say $P_1 = s_1, ..., 2, u, 3, ..., r_1$. Then we can let $\bar{P}_1 = s_1, ..., 2, 3, ..., r_1$ and $\bar{P}_2 = P_2$ and $\bar{P}_3 = P_3$ and $\bar{P}_4 = 1, u, 4$. Thus we can find four point disjoint 1-4 paths not containing edge 14, and again the minimality of H is contradicted.

THEOREM 13. *The class of* 4-*connected graphs is the class of graphs obtained from* K_5 *by finite sequences of line addition,* 4-*soldering,* 4-*point-splitting,* 4-*line-splitting, and* 3-*fold*–4-*point-splitting.*

Proof. By Theorem 0, Theorem 1, Theorem 2, Theorem 9 and Theorem 12 one stays in the class of 4-connected graphs. To show that one obtains all of the 4-connected graphs by these sequences, apply induction on the number of points. Let H be a 4-connected graph with at least six points. Since line addition is allowed, assume that H is minimally-4-connected. Now H has a point u of degree four. If u is from Case 1–8, let G be the 4-connected graph obtained from H by Theorem 4. Since G has one fewer points we can obtain G by such a sequence, and H is obtained from G by 4-soldering. If u is from Case 10 or 11 let G be the appropriate 4-connected H_{iu}, where $1 \leqslant i \leqslant 4$, which is guaranteed by Theorem 10 and Theorem 11. Since G has one fewer point we can obtain G, and H is obtained from G by 4-point-splitting. Finally suppose that every point in H of degree four is as in Case 9. By Lemma 5 either point 4 or point 1 is of degree four, say 4 is such a point. Now 4 is also a Case 9 point. Hence if deg $1 \geqslant 5$ use Lemma 4 to obtain the 4-connected graph G with one fewer points. H can be obtained from G by 4-line-splitting. If deg $1 = 4$ then 1 is also a Case 9 point, so we can use Lemma 3 to obtain the 4-connected graph G with two fewer points. H can be obtained from G by 3-fold–4-point-splitting.

4. OBSERVATIONS

It should be noted that *I* used minimality of 4-connected graphs, with one important exception, only to guarantee a point of degree four. Better results can be obtained. For example, if u is a Case 7 point in a minimally-4-connected graph H then Theorem 4 tells us that G is 4-connected. In fact, G minus edge 14 is also 4-connected [4].

It appears as if classifying 4-connected graphs is only a beginning in exploiting the concepts of "soldering" and "splitting." I believe that n-connected graphs will be classified by "soldering" and "generalized splitting" operations. Further results along these lines will appear in [4].

REFERENCES

1. R. HALIN, A theorem on n-connected graphs, *J. Combinatorial Theory* **7** (1969), 150–154.
2. F. HARARY, "Graph Theory," Addison-Wesley, Reading, MA, 1969.

3. S. Hedetniemi, Characterizations and constructions of minimally 2-connected graphs and minimally strong digraphs, "Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing," pp. 257–282, (1971).
4. P. J. Slater, Classifying n-connected graphs, in preparation.
5. W. T. Tutte, "Connectivity in Graphs," Mathematical Expositions, No. 15, University of Toronto Press, Toronto, 1966.
6. D. Barnette, Generating planar 4-connected graphs, *Israel J. Math.* **14** (1973), 1–13.